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On the origin of the Schwinger anomaly

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Abstract. For a simple non-relativistic fermion model we show that the Schwinger anomaly can be viewed as an effect of the infinite depth of the Dirac sea.

It is now well known [1-3] that the Dirac sea may be viewed as an origin of the Schwinger terms in the commutators of currents. Actually, it is the infinite depth of the Dirac sea that proves to be the source of anomalous terms. Mathematically, this phenomenon is quite natural because filling in the Dirac sea implies the transition to be a non-equivalent representation of canonical anticommutation relations. The aim of the present paper is to study all this machinery in detail. My interest in this field was stimulated by some problems in quantum integrable systems where filling in the Dirac sea is also claimed to produce Schwinger terms [4].

We choose to deal with the simplest model where the phenomenon yet exists. Consider a free non-relativistic one-dimensional model of the one-component fermion field

$$\psi(x)=\frac{1}{\sqrt{2\pi}}\int \mathrm{d}k\, \mathrm{e}^{\mathrm{i}kx}a_k. \tag{1}$$

The momenta could even be discrete, which corresponds to a finite box in x space. However, I prefer to use continuous momenta because the discretisation does not lead to any simplification. The anticommutation relations are standard canonical ones:

$$[a_k, a^+_{k'}]_+=\delta(k-k') \tag{2}$$

$$[\psi(x), \psi^+(y)]_+=\delta(x-y). \tag{3}$$

To be a free one, our model must have the Hamiltonian which is bilinear in the fields or in the a, a^+ operators. For our purpose it is not even necessary to fix the Hamiltonian unambiguously. Let it be

$$H=\int \mathrm{d}k\, a^+_k a_k \omega(k) \tag{4}$$

with a function $\omega(k)$ such that in a certain interval $\omega(k)<0$. We fix this interval to be $-\Lambda<k<0$. Just here the Dirac sea will reside. Λ is a positive number which eventually will go to $+\infty$.

Anticipating the result of our investigation we can formulate the final answer: the Schwinger terms appear if the interval of negative values of $\omega(k)$ is infinite ($\Lambda=\infty$) or, in the discrete version $H=\Sigma_k\, \omega_k a^+_k a_k$, if the number of k with ω_k negative is infinite. Both cases can be interpeted as the presence of an infinitely deep Dirac sea to fill in.

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It should be emphasised that the magnitude of $\omega(k)$ is not relevant; only the sign of it is.

Now let us describe the Fock space of the model. The vacuum $|0\rangle$ of the operators a, a^+,

$$a_k|0\rangle = 0 \tag{5}$$

is a mathematical or false vacuum because it is not the state of the lowest energy. The true or physical vacuum $\|0\rangle$ emerges as a result of filling in the Dirac sea. In other words, $\|0\rangle$ is the vacuum of new operators b, b^+,

$$b_k\|0\rangle = 0 \tag{6}$$

defined by

$$\begin{aligned} a_k &= \theta_\Lambda(k)b_k + (1-\theta_\Lambda(k))b_k^+ \\ a_k^+ &= \theta_\Lambda(k)b_k^+ + (1-\theta_\Lambda(k))b_k \end{aligned} \tag{7}$$

where $\theta_\Lambda(k)$ generalises the ordinary θ function:

$$\theta_\Lambda(k) = \begin{cases} 1 & \text{for } k>0 \\ 0 & \text{for } -\Lambda<k<0 \\ 1 & \text{for } k<-\Lambda \end{cases} \tag{8}$$

so that

$$\theta_\Lambda(k) \xrightarrow[\Lambda\to\infty]{} \theta(k). \tag{9}$$

The operators b, b^+ obey the canonical anticommutation relations

$$[b_k, b_{k'}^+]_+ = \delta(k-k'). \tag{10}$$

In terms of b, b^+ the Hamiltonain H is positive definite.

In what follows we shall study various operators in this new Fock space over the vacuum $\|0\rangle$. The operators a, a^+ when encountered are to be substituted by (7) with Λ finite or infinite.

Consider the current (or particle density)

$$I(x) = \psi^+(x)\psi(x) \tag{11}$$

and its commutator

$$[I(x), I(y)]_- = ? \tag{12}$$

Naively, due to (3), the latter seems to be equal to zero. But we shall see that for $\Lambda=\infty$ the answer will be quite different: the Schwinger term will appear. How does this happen?

Let us begin by considering the Fourier transform of $I(x)$:

$$I(p) = \int_{-\infty}^{\infty} \mathrm{d}k\, a_{k+p}^+ a_k. \tag{13}$$

By our convention, we must treat a, a^+ in terms of b, b^+ using (7). Rewritten in this way, $I(p)$ is to be denoted by $I_\Lambda(p)$ and reads

$$\begin{aligned} I_\Lambda(p) = \int_{-\infty}^{\infty} \mathrm{d}k[&\theta_\Lambda(k+p)\theta_\Lambda(k)b_{k+p}^+b_k + \theta_\Lambda(k+p)(1-\theta_\Lambda(k))b_{k+p}^+b_k^+ \\ &+(1-\theta_\Lambda(k+p))\theta_\Lambda(k)b_{k+p}b_k \\ &+(1-\theta_\Lambda(k+p))(1-\theta_\Lambda(k))b_{k+p}b_k^+]. \end{aligned} \tag{14}$$

Two points should be made here. Firstly, for different values of Λ, $I_\Lambda(p)$ become different operators in the same Fock space. Secondly, $I_\Lambda(p)$ is not normal-ordered with respect to the b. It consists of four terms, and one of them is not ordered. However, operators similar to (13), when non-ordered, can prove not to be the true operators in the Fock space. This is due to the infinite interval of k integration. For example, $\int_{-\infty}^{0} \mathrm{d}k\, b_k b_k^+$ is not the true Fock space operator.

Anticipating the limit $\Lambda \to \infty$, we have to deal with normal-ordered operators. Let us introduce the notation $:I: \equiv J$ (the ordering with respect to b-operators, of course). It is easy to obtain (for finite Λ)

$$I_\Lambda(p) = J_\Lambda(p) + \Lambda\delta(p) \tag{15}$$

$$[I_\Lambda(p), I_\Lambda(p')]_- = [J_\Lambda(p), J_\Lambda(p')]_- = 0. \tag{16}$$

The derivation of (16) is straightforward. The terms with operators in the normal form cancel after a trivial shift of an integration variable. The remaining c-number term is equal to

$$\begin{aligned} \delta(p+p') \int_{-\infty}^{\infty} & \mathrm{d}k\,\mathrm{d}q[\delta(k-q-p')\delta(k+p-q) - \delta(k-q)\delta(k+p-q-p')] \\ & \times[(1-\theta_\Lambda(k+p))\theta_\Lambda(k)\theta(q+p')(1-\theta_\Lambda(q)) \\ & - \theta_\Lambda(k+p)(1-\theta_\Lambda(k))(1-\theta_\Lambda(q+p'))\theta_\Lambda(q)] \\ & = \delta(p+p') \int_{-\infty}^{\infty} \mathrm{d}k[\theta_\Lambda(k)(1-\theta_\Lambda(k+p)) \\ & - \theta_\Lambda(k+p)(1-\theta_\Lambda(k))] = 0. \end{aligned} \tag{17}$$

Up to now, no Schwinger terms exist.

However, the last line of (17) hints that for $\Lambda = \infty$ the result could be non-zero. Consider the limit $\Lambda \to \infty$ in detail. $I_\Lambda(p)$ does not survive in this limit; it diverges due to the $\Lambda\delta(p)$ term, whereas $\lim_{\Lambda\to\infty} J_\Lambda(p)$ is evidently a well defined operator. To find it explicitly, one should replace the a in $I(p)$, $I(p')$ by (7) using ordinary θ functions and then perform the normal ordering of the b. The result is

$$\begin{aligned} \lim_{\Lambda\to\infty} J_\Lambda(p) \equiv J_\infty(p) = \int_{-\infty}^{\infty} & \mathrm{d}k[\theta(k)\theta(k+p)b_{k+p}^+ b_k + \theta(k+p)\theta(-k)b_{k+p}^+ b_k^+ \\ & + \theta(k)\theta(-k-p)b_{k+p}b_k - \theta(-k)\theta(-k-p)b_k^+ b_{k+p}]. \end{aligned} \tag{18}$$

One can straightforwardly show that [2]

$$[J_\infty(p), J_\infty(p')]_- = -p\delta(p+p') \tag{19}$$

which corresponds to

$$[J(x), J(y)]_- = \frac{\partial}{\partial x}\delta(x-y) \tag{20}$$

the RHS being the Schwinger term.

Returning to (19) we observe that

$$[\lim_{\Lambda\to\infty} J_\Lambda(p), \lim_{\Lambda\to\infty} J_\Lambda(p')]_- = -p\delta(p+p') \neq 0 \tag{21}$$

whereas

$$\lim_{\Lambda\to\infty} [J_\Lambda(p), J_\Lambda(p')]_- = 0. \tag{22}$$

So, taking the product and taking the limit does not commute. This is possible since the difference

$$J_\infty(p) - J_\Lambda(p) \tag{23}$$

does not vanish in the operator sense as $\Lambda \to \infty$. In fact, this difference is a sum of several terms, typical of which are

$$\int_{-\infty}^{-\Lambda} \mathrm{d}k\, b_{k+p}^{+} b_k \qquad \int_{-\Lambda-p}^{-\Lambda} \mathrm{d}k\, b_{k+p}^{+} b_k^{+} \qquad \text{etc.} \tag{24}$$

All these terms are of the form $\int^{-\Lambda} \mathrm{d}k\, \hat{A}(k, p)$ with a bilinear operator $\hat{A}$ and therefore vanish in the limit $\Lambda \to \infty$ when sandwiched between two given Fock states. However, the norm of an operator like (24) does not go to zero as $\Lambda \to \infty$. We can conclude that the mathematical nature of the Schwinger anomaly consists of the fact that J_Λ does not converge to J_∞ in the operator sense when the depth of the Dirac sea goes to infinity.

Perhaps the most clarifying formula is

$$J_\infty(p) = I_\Lambda(p) - \Lambda\delta(p) + \int_{-\infty}^{-\Lambda} \mathrm{d}k\, \hat{A}(k, p). \tag{25}$$

We see that the 'anomalous' current $J_\infty(p)$ differs from the 'commuting' one $I_\Lambda(p)$ not by a mere c-number term $\Lambda\delta(p)$ as it would if $I_\infty(p)$ could exist, but also by a certain operator term (the last term in the RHS of (25)). It is just this term that generates Schwinger anomalies. It resides deep in the Dirac sea and, as $\Lambda \to \infty$, seems to be drowned. However, the c-number commutators produced by the $\hat{A}$ terms do not depend on Λ at all, do not drown, and eventually display themselves in the form of anomalies.

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